

Equivalent Cake Filtration Model

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Abstract Cake filtration has been widely used in many chemical processes with more non-Newtonian, highly viscous and compressible materials involved. Neither traditional nor modern filtration theory can be applied in practice. “Equivalent cake filtration model” is a recently developed mathematical model to describe cake filtration for both Newtonian and non-Newtonian fluids, in either steady or unsteady filtration stages. This model has two strengths: (1) It can be used to determine equivalent capillary radii and predict filtration quality based on the properties of solid/liquid system and operation parameters; and (2) to calculate cake specific resistance and its variations with time at various cake thickness locations.

Keywords cake filtration, equivalent cake filtration model, specific cake resistance, filtrate rate, filtrate quality, equivalent capillary radii, non-Newtonian filtration

1 INTRODUCTION

Cake filtration is one of the most common and important unit operations in chemical engineering process. Cake filtration model provides a theoretical basis for process design, equipment selection, scale magnification, performance prediction, and structural optimization. Traditional theories by Karman and Ruth [1] focused on average cake resistance. However, these theories did not cover the non-Newtonian slurry. Modern filtration theory was given by Tiller and Shirato [2, 3], and Rushton [4] formed the local specific resistance and cake compressibility concepts, but these were tough for practical use. Similar work included the capillary net model by Dullien and Sorbie [5, 6], and cake geometry study by Xu [7] and Li [8].

In relation to the non-Newtonian fluid filtration, the most accepted model was formed by Kozicki and Shirato, based on Darcy's Law [3] as follows:

$$\left(\frac{dv}{dt} \frac{1}{A}\right)^n = \frac{1}{\mu \alpha C_s} \frac{d\Delta p}{dl} \quad (1)$$

where dv/dt is the flow rate ($\text{m}^3 \cdot \text{s}^{-1}$), A is the cake area (m^2), n is non-Newtonian index, μ is viscosity ($\text{Pa} \cdot \text{s}$), α is Ruth's specific resistance (m^2), C_s is the dry solids mass per unit filtrate mass, Δp is the differential pressure (Pa), and l is the cake thickness (m).

Equation (1) is theoretically sound. However, the specific cake resistance is not obtained because of the complexity and dynamic variations of the filter cake structure. In addition, it is difficult to measure or calculate the viscosity μ of non-Newtonian fluid.

This article focuses on predicting filtration quality and calculating cake specific resistance based on operational parameters.

2 EQUIVALENT MODEL

2.1 General description

The cake filtration is typically an unsteady proc-

ess. When filtration proceeds, effective filtering area and related flow rate decreases continuously. Besides, flow channels are plugged gradually and new ones form. Moreover, average specific resistance increases, and possibility that impurities flow through the cake decline. These features are mathematically indescribable, but can be physically described as follows [9]:

(1) All capillary radii in an equivalent cake reduce and their lengths prolong synchronously with the filtration process, so the cake unsteady features are easily described, such as the amendment of filtration and augment of resistance.

(2) Cake aging features, like the deposition of the impurity and the shrinking of the effective area, are “equivalent” to reduction in capillary radius and the increase in length. Different time and different radii corresponds to different flow rates and specific resistances, respectively. Whereas different cake locations with the same equivalent radius correspond to different specific resistances as the cake grows to a certain thickness. The macroscopic equivalent is of more importance when compared to microscopic similarity, in general.

Equivalent cake filtration model assumes that the liquid uniformly flows through a bundle of capillaries under the pressure drop Δp , and the flow rate through the capillaries is the same as that through a real filter cake. Thus, the capillaries bundle is called “the equivalent cake”, and radii of the capillaries are called “equivalent radii”.

2.2 Basic assumptions

(1) The amount of the capillaries that constitute the cake is N , without fluctuation, and there is no capillary clogging or plugging during the filtration-cyclical process.

(2) All capillaries are cylindrical and straight with the same radius $R(t)$, which reduces as filtration proceeds. Moreover, flow rate $q(t)$ through each capillary is invariable.

(3) The length of each capillary $l(t)$ is equal to the

cake thickness, which varies as filtration proceeds.

(4) Flow rate of the liquid flowing through the simulated filter cake is similar to that of actual filter cake when pressure drop Δp is set.

During a real filtration process, filtrate flow channels gradually dwindle with continuous reduction in flow rate and increase in resistance $\alpha(t)$. Correspondingly, in an equivalent cake, the decrease in capillary radii $R(t)$, the increase in cake specific resistance $\alpha(t)$, and the decrease in filtrate rate takes place with the filtration in progress. The reduction of $R(t)$ during the process provides a key to solve unsteady and dynamic filtration problem.

3 NON-NEWTONIAN CAPILLARY FLOW

Navier-Stokes equation (N-S) [10] is used to form capillary flow equations, with viscosity as a constant for Newtonian flow but as a variable for non-Newtonian flow. A non-Newtonian capillary flow model (Fig. 1) with following assumptions is provided:

- (1) Round and straight capillary with $\Delta p = p_1 - p_2$ between two ends;
- (2) The value of the Reynolds number is low, and laminar flow only exists in direction z with no rotation around θ and however, all the inertia and weight forces are ignored;
- (3) Fluids are incompressible, and $\nabla \mathbf{V} = 0$, where $\mathbf{V}$ is flow speed tensor;
- (4) The flow is steady;
- (5) Non-Newtonian flow is governed by Ostwald-de Waele's power law [7]:

$$\tau = M \dot{\gamma}^n \quad (2)$$

where M is viscosity coefficient, τ is shear stress (Pa), and $\dot{\gamma}$ is shear rate (s^{-1}).

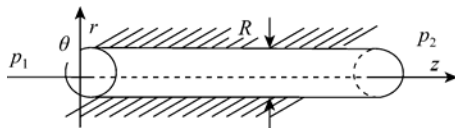


Figure 1 The capillary flow model

In Fig. 1, all velocity components are zero except in direction z , i.e., $V_\theta = 0$, $V_r = 0$. Thus, the N-S equation becomes

$$\rho \frac{dV_z}{dt} = \mu \nabla^2 V_z + \mu \nabla \cdot \mathbf{V} + \rho g_z - \frac{\partial p}{\partial z}, \quad (3)$$

where ρ is mass per unit volume ($\text{kg} \cdot \text{m}^{-3}$).

In non-Newtonian flow, μ is not a constant in the Laplacian. Incorporating the assumptions, Eq. (3) can be rewritten as

$$\frac{\partial p}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial V_z}{\partial r} \right), \quad (4)$$

where r is variable radius, V_z is the velocity component in direction z , and $\partial p / \partial z = \Delta p / l$ is the pressure gradient.

According to the definition of viscosity, one obtains

$$\mu = \tau / \dot{\gamma} \quad \text{and} \quad \dot{\gamma} = -\partial V_z / \partial r, \quad (5)$$

where V_z decreases with increase in r , thus, there is a minus sign in the equation. Because filtration flow always locates in low shear stress area, there must be τ_0 , which corresponds to a point where $\dot{\gamma}_0 = 1 \text{ s}^{-1}$, or $\tau_0 = \mu_0 \dot{\gamma}_0 = \mu_0$. Therefore,

$$\frac{\dot{\gamma}}{\dot{\gamma}_0} = \left(\frac{\tau}{\tau_0} \right)^{1/n} = \left(\frac{\mu}{\mu_0} \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{1/n} \quad \text{i.e.,} \quad \mu = \mu_0 \dot{\gamma}^{n-1}. \quad (6)$$

Substituting Eqs. (5) and (6) into Eq. (4), then obtains

$$-\frac{\partial (r \mu_0 \dot{\gamma}^n)}{\partial r} = \frac{r \Delta p}{l} \quad (6-1)$$

The above equation can be integrated under the following boundary conditions:

(1) At the center of the capillary $r = 0$, and $\partial V_z / \partial r = 0$.

(2) At the internal surface of the capillary $r = R$, and $V_z = 0$.

From the definition of capillary flow rate,

$$q = \int_0^R V_z(r) \cdot 2\pi r \cdot dr, \quad \text{one gets}$$

$$q = \frac{n}{3n+1} \left(\frac{\Delta p R}{2\mu_0 l} \right)^{1/n} \pi R^3. \quad (7)$$

Equation (7) can be applied to non-Newtonian flow when μ_0 corresponds to the point where $\dot{\gamma} = 1 \text{ s}^{-1}$.

4 EQUIVALENT FLOW RATE

On the basis of the assumption of equivalent cake filtration model, the flow rate of liquid flowing through the capillaries is the same as that of the actual filter cake under the same pressure drop Δp at different time:

$$Q(t) = Nq(t), \quad (8)$$

where $Q(t)$ is flow rate through the real cake, which can be determined by experiments; N is the number of total capillary tubes, and $q(t)$ is the rate flowing through a single capillary.

Substituting Eq. (7) into Eq. (8), one gets

$$\begin{aligned} Q(t) &= Nq(t) \\ &= \frac{nN}{3n+1} \left[\frac{\Delta p(t) R(t)}{2\mu_0 l(t)} \right]^{1/n} \pi R(t)^3 \\ &= \frac{A_c(t) N n}{3n+1} \left[\frac{\Delta p(t)}{2\mu_0 l(t)} \right]^{1/n} R(t)^{\frac{n+1}{n}} \end{aligned} \quad (9)$$

where $R(t)$ is the equivalent radii of capillary tubes, and $A_c(t)$ is the cross section area of each capillary tube. If the total filtration area is A , then the porosity of equivalent cake is equal to that of real cake and one obtains

$$\frac{A_c(t)}{A} = \varepsilon(t) \quad \text{or} \quad A_c(t) = A \varepsilon(t). \quad (10)$$

Substituting Eq. (10) into Eq. (9) yields

$$Q(t) = Nq(t) = \frac{A\varepsilon(t)n}{3n+1} \left[\frac{\Delta p}{2\mu_0 l(t)} \right]^{\frac{1}{n}} R(t)^{\frac{n+1}{n}}, \quad (11)$$

where $\alpha(t)$ is porosity of cake, and can be determined by experiments. Eq. (11) can be rewritten as

$$R(t) = \left[\frac{Q(t)}{A} \frac{3n+1}{n\varepsilon(t)} \right]^{\frac{n}{n+1}} \left[\frac{2\mu_0 l(t)}{\Delta p} \right]^{\frac{1}{n+1}}. \quad (12)$$

During an unsteady filtration process, the flow rate continuously decreases, which corresponds to the decrease in $R(t)$. Eq. (7) shows steady flow, and the deduction of Eqs. (11) and (12) based on Eq. (7) by introducing time t provides a way to describe the unsteady filtrate flow. During filtration process, the $R(t)$ decreases as $Q(t)$ decays, which is related to the decrease in the actual cake pore size during a real filtration process. Furthermore, Eq. (10) indicates that

(1) When other conditions are fixed; the lower μ_0 is the $R(t)$ also decreases. Ways to decrease $R(t)$ include temperature control, solvent components optimization, and application of reagents to increase solubility.

(2) When other conditions are fixed; the lower the filtration pressure is the higher is the filtrate quality.

(3) The relation between porosity, Δp , and flow rate on filtrate quality is quite complicated. Eq. (1) can assist in the selection of optimized operating parameters to achieve the best filtrate quality.

(4) Porosity $\varepsilon(t)$ depends on particle's physical and chemical properties in suspensions, and the manner of the particle deposition. Furthermore, $\varepsilon(t)$ is related to $R(t)$.

Though equivalent cake filtration model assumes that the capillary is round and straight, but the actual situation is different. Pore size, shape and length are variable, and inlet Bagley phenomenon and outlet streamline diffusion [6, 10] are not considered. However, the model and calculation of $R(t)$ can still serve as an effective tool to predict filtration flow rate and quality variation during the dynamic filtration process.

5 CAKE SPECIFIC RESISTANCE CALCULATION

With the same pressure drop, filtrate flow rate of the equivalent cake is equivalent to that of an actual one, thus, they should share the same cake specific resistance based on Eq. (11), which can be simplified to the following form:

$$\left(\frac{Q}{A} \right)^n = \left[\frac{1}{2} \left(\frac{n\varepsilon}{3n+1} \right)^n R^{n+1} \right] \frac{1}{\mu_0} \frac{\Delta p}{l}. \quad (13)$$

Combining Eq. (13) with Eq. (1) and Darcy's law, cake permeability K can be deducted as

$$K = \frac{1}{2} \left(\frac{n\varepsilon}{3n+1} \right) R^{n+1}. \quad (14)$$

Substituting $\alpha = (KC_s)^{-1}$ into Eq. (13), gets:

$$\alpha = \left(\frac{3n+1}{n\varepsilon} \right)^n \frac{2}{R^{n+1} C_s}. \quad (15)$$

Substitution of Eq. (15) into Eq. (13) yields

$$\left(\frac{Q}{A} \right)^n = \frac{1}{\alpha C_s \mu_0} \frac{\Delta p}{l}. \quad (16)$$

Formation of Eq. (16) is basically identical to Eq. (1) except that viscosity μ_0 is replaced by μ in Eq. (1). μ_0 is obtained with a accurate rheometric test of the slurry. On the contrary, it is almost impossible to calculate μ in Eq. (1). The introduction of μ_0 makes Eq. (1) effective in application.

For Newtonian fluid, when $n=1$, and $\mu = \mu_0$, Eq. (15) can be simplified to

$$\alpha(t) = \frac{8}{\varepsilon(t) R(t)^2 C_s}. \quad (17)$$

Equations (15) and (17) provide a method to calculate local and average cake specific resistance when porosity $\varepsilon(t)$ can be measured, and sometimes obtained by liquid material balance equation. From the two equations above, it can be concluded that

(1) For Newtonian fluid, the specific cake resistance is only related to particle deposition and cake structure parameters such as $\varepsilon(t)$ and $R(t)$, and does not relate to fluid properties. Whereas in non-Newtonian fluid, both cake structure parameters and fluid factor n are involved; this is considered the main difference between the two fluids.

(2) For both Newtonian and non-Newtonian slurry, $\alpha(t)$ increases as $R(t)$ decreases.

(3) Specific cake resistance $\alpha(t)$ rises greatly as n becomes larger continuously, when the slurry changes from shear thinning to shear thickening.

6 EXAMPLE

Table 1 gives the tested filtration data, such as filtrate rate, cake moisture content, particular r_{50} in filtrate, and calculated data, such as $R(t)$ and specific cake resistance based on Eqs. (12) and (15). Fig. 2 shows correlations of $R(t)$, r_{50} , and press life. The test is under a constant Δp of 0.4 MPa, with dope viscosity of 62 Pa·s at 50°C and $n=0.60$. Liquid in cake can be used as a measure of porosity based on material balance.

Table 1 Calculation and test results

Press life /d	Rate /kg·h ⁻¹	Cake moisture /%	Filtrate r_{50} /μm	$R(t)$	$\alpha(t)$ /m ⁻²
11	765	59.79	2.05	6.8	1.25×10 ¹²
21	484	56.63	1.97	6.3	1.47×10 ¹²
31	313	54.01	1.95	6.2	1.56×10 ¹²
42	172	52.44	1.90	5.3	2.05×10 ¹²
51	125	50.72	1.88	5.0	2.29×10 ¹²
54	105	49.95	1.87	4.8	2.32×10 ¹²

Note: Plate and frame presses were used with a filtration area of 50 m² each, a maximal cake thickness of 14 mm, and a differential pressure of 0.4 MPa. Particular r_{50} in filtrate used to characterize filtrate purity is analyzed by laser particle size analyzer (HRLD—150JA, HIAC/ROYCO). Press life was just 54 days long.

A comparison is shown in Table 2 before and after corrective actions according to Eq. (12), including larger filtration area, less filtrate recycle rate, and higher dope temperature. Extrusion stability is used as an important parameter of filtration quality. From Tables 1, 2 and Fig. 2, we have the following discussions:

(1) Specific cake resistances go up sharply, whereas flow rate decreases with time as the cake ages.

(2) Both the $R(t)$ and particular radii in filtrate (impurity radii) decrease synchronously as the filtration process proceeds.

(3) $R(t)$ are 2.8–3.3 times larger than filtrate, particular r_{50} , although $R(t)$ and r_{50} are absolutely two different issues from different sources. The error may be caused by the hypotheses made for the model and complexity of cake filtration. Compared with other models, the error is not too bad.

(4) It is easier to use cake moisture instead of porosity, but may cause problems because of an idle void existing inside the cake. $R(t)$ may be underestimated and resistance overrated, although the error seems to be acceptable.

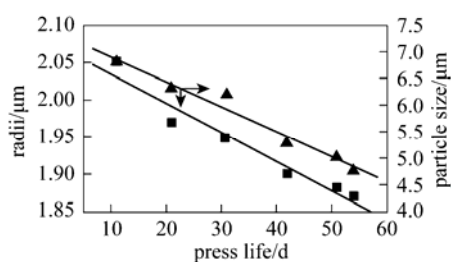


Figure 2 Equivalent radii and average impurity size

Table 2 Filtration quality improvement

	Filtrate $r_{50}/\mu\text{m}$	$R(t)$	Extrusion stability/times $\cdot\text{t}^{-1}$
before	2.41	10.8	0.61
after	1.95	6.2	0.39

Filtration is a very complicated process, in which both solids and liquid move, filter channels deform, filtration effect change, and huge difficulties begin when one tries to model it. This is why a precise mathematical description for cake filtration does not exist till date. Therefore, the model focuses on macroscopic equivalence in filtration process and not microscopic similarity. Nevertheless, relatively strong hypotheses made in this article is not an obstacle in terms of the equivalence of flow rate as expressed in Eq. (8), although further work needs to be done.

7 CONCLUSIONS

Equivalent concept is used to model cake filtra-

tion. $R(t)$ and cake resistances can be easily calculated based on filtration parameters by Eqs. (12) and (15). $R(t)$ could be used as an indicator of filtration quality, and $\alpha(t)$ could be used as a prediction of filtrate rate for both Newtonian and non-Newtonian fluids. Unsteady features of cake filtration are finitely described.

NOMENCLATURE

A	cake filtration area, m^2
$A_0(t)$	cross section area of each capillary tube at time t , m^2
C_s	dry solids mass per unit filtrate mass
dv/dt	flow rate, $\text{m}^3\cdot\text{s}^{-1}$
l	cake thickness, m
$l(t)$	cake thickness at time t , m
M	viscosity coefficient
N	amount of total capillary tubes
n	non-Newtonian index
Δp	differential pressure, MPa
$Q(t)$	flow rate through the real cake, $\text{m}^3\cdot\text{s}^{-1}$
$q(t)$	flow rate of a single capillary at time t , m^3/s^{-1}
$R(t)$	equivalent radii of capillary tubes at time t , m
r	variable radius, m
t	time
V	flow speed tensor, $\text{m}\cdot\text{s}^{-1}$
V_z	the component in direction z , $\text{m}\cdot\text{s}^{-1}$
z	flow direction
α	Ruth's specific resistance, m^{-2}
$\dot{\gamma}$	shear rate, s^{-1}
$\varepsilon(t)$	porosity of filter cakes at time t
μ	viscosity, Pa $\cdot\text{s}$
μ_0	a special viscosity point when $\dot{\gamma} = -1$, Pa $\cdot\text{s}$
ρ	weight per unit volume, $\text{kg}\cdot\text{m}^{-3}$
τ	shear stress, Pa

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